Question ①

(Q-1) Let $\{f_k\}_{k=1}^{m}$ be a frame for an n-dimensional vector space V and let B denote the optimal upper bound. Prove

$$B \leq \sum_{k=1}^{m} \|f_k\|^2 \leq nB.$$

Soln:- Let $\{e_1, e_2, \ldots, e_n\}$ be an O.N.B of V.

Then $f_k = \alpha_1^k e_1 + \cdots + \alpha_n^k e_n$, for $k \in \{1, \ldots, m\}$

Now we compute, for any $i \in \{1, \ldots, n\}$

$$\sum_{k=1}^{m} |\langle e_i, f_k \rangle|^2 = \sum_{k=1}^{m} |\langle e_i, \alpha_1^k e_1 + \cdots + \alpha_n^k e_n \rangle|^2$$

$$= \sum_{k=1}^{m} |\langle e_i, \alpha_i^k e_i \rangle|^2$$

$$= \sum_{k=1}^{m} |\alpha_i^k|^2$$

Using the fact that B is optimal upper bound, we have

$$\sum_{k=1}^{m} |\alpha_i^k|^2 = \sum_{k=1}^{m} |\langle e_i, f_k \rangle|^2 \leq B\|e_i\|^2 = B, \quad \forall i \in \{1, \ldots, n\}$$

$$\therefore \sum_{i=1}^{n} \left(\sum_{k=1}^{m} |\alpha_i^k|^2 \right) \leq nB. \quad \text{———} (*)$$

Now $$\sum_{k=1}^{m} \|f_k\|^2 = \sum_{k=1}^{m} \|\alpha_1^k e_1 + \cdots + \alpha_n^k e_n\|^2$$

$$= \sum_{k=1}^{m} \left(\sum_{i=1}^{n} |\alpha_i^k|^2 \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} |\alpha_i^k|^2$$

$$\leq nB \quad (\text{By } (*))$$

$\because$ B is optimal upper bound and $\sum_{k=1}^{m} \|f_k\|^2$ is one of the choice of upper bound for the frame $\{f_k\}_{k=1}^{m}$,

$$\therefore \quad B \le \sum_{k=1}^{m} \|f_k\|^2$$

$$\therefore B \le \sum_{k=1}^{m} \|f_k\|^2 \le nB.$$

□

(Q-2) Show that a frame for $\mathbb{R}^n$ is also a frame for $\mathbb{C}^n$.

Sol^n Let $\{f_j\}_{j=1}^{m}$ be a frame for $\mathbb{R}^n$ with bounds A & B. — (1)

Then $f_j = (\alpha_1^j, \ldots, \alpha_n^j)$

Let $f = (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n) \in \mathbb{C}^n$ be arbitrary, where $\alpha_i, \beta_i \in \mathbb{R}$, $\forall\ i = 1, \ldots, n$.

We compute

$$\sum_{j=1}^{m} |\langle f, f_j \rangle|^2 = \sum_{j=1}^{m} |\langle (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n), (\alpha_1^j, \ldots, \alpha_n^j) \rangle|^2$$

$$= \sum_{j=1}^{m} |(\alpha_1 + i\beta_1)\alpha_1^j + \ldots + (\alpha_n + i\beta_n)\alpha_n^j|^2$$

$$= \sum_{j=1}^{m} |(\alpha_1\alpha_1^j + \ldots + \alpha_n\alpha_n^j) + i(\beta_1\alpha_1^j + \ldots + \beta_n\alpha_n^j)|^2$$

$$= \sum_{j=1}^{m} \left(|\alpha_1\alpha_1^j + \ldots + \alpha_n\alpha_n^j|^2 + |\beta_1\alpha_1^j + \ldots + \beta_n\alpha_n^j|^2\right)$$

$$\left[\because |\alpha + i\beta| = \sqrt{|\alpha|^2 + |\beta|^2}\right]$$

$$= \sum_{j=1}^{m} \left(|\langle (\alpha_1, \ldots, \alpha_n), (\alpha_1^j, \ldots, \alpha_n^j) \rangle|^2 + |\langle (\beta_1, \ldots, \beta_n), (\alpha_1^j, \ldots, \alpha_n^j) \rangle|^2\right)$$

$$= \sum_{j=1}^{m} |\langle (\alpha_1, \ldots, \alpha_n), f_j \rangle|^2 + \sum_{j=1}^{m} |\langle (\beta_1, \ldots, \beta_n), f_j \rangle|^2$$

— (2)

Now B is optimal upper bound and $\sum_{k=1}^{m}\|f_k\|^2$ is one of the choice of upper bound for the frame $\{f_k\}_{k=1}^{m}$

$$\therefore\ B \le \sum_{k=1}^{m}\|f_k\|^2$$

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Eg Show that a seq. $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a Bessel seq. if either of the following two conditions holds

(a) $\sum_{m}^{\infty}\sum_{n}^{\infty}|\langle f_m, f_n\rangle|^2 < \infty$

OR

(b) $\sup_{m}\sum_{n=1}^{\infty}|\langle f_m, f_n\rangle|^2 < \infty$

Sol Let $K = \sum_{m}^{\infty}\sum_{n}^{\infty}|\langle f_m, f_n\rangle|^2 < \infty$. and let $\mathcal{G} = [\langle f_m, f_n\rangle]_{m,n}^{\infty}$ be the Gram matrix associated with $\{f_k\}_{k=1}^{\infty}$. Then, for any $c = \{c_k\}_{k}^{\infty} \subset C_{00}$, we have

$$\|\mathcal{G}c\|^2 = \sum_{m}^{\infty}\left|\sum_{n}^{\infty}\langle f_m, f_n\rangle c_n\right|^2$$

$$\le \sum_{m}^{\infty}\left(\sum_{n}^{\infty}|\langle f_m, f_n\rangle|^2\right)\left(\sum_{n}^{\infty}|c_n|^2\right)$$

$\leq K \,\|\{c_k\}_{k}^{\infty}\|^2$

$\Rightarrow$ that the Gram-matrix G defines a bdd linear operator on ℓ^2. Hence, $\{f_k\}_k^{\infty}$ is a Bessel seq.

(b) Hint: compute.

$$\|Gc\|^2 = \sum_m^{\infty} \Big| \sum_n^{\infty} \langle f_m, f_n \rangle c_m \Big|^2$$

$$\leq$$

$$\vdots$$

$$\leq K^2 \,\|\{c_k\}_{k}^{\infty}\|$$

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Theorem: Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ be a Bessel sequence with Bessel bound B. Then, $\exists$ a seq. $\{\phi_j\}_{j\in J}$ in $\mathcal{H}$ s.t.

$$\{f_k\}_{k=1}^{\infty} \cup \{\phi_j\}_{j\in J}$$

is a tight frame for $\mathcal{H}$ with frame bound B.

Proof: Let $S: \mathcal{H} \to \mathcal{H}$ be the frame operator associated with $\{f_k\}_{k=1}^{\infty}$, and I be the identity operator on $\mathcal{H}$.

Then $BI - S$ is a self-adjoint and positive operator.

Let $(BI-S)^{\frac{1}{2}}$ be the square root of $(BI-S)$. Then, we can write.

$$Bf = Sf + (BI-S)^{\frac{1}{2}}(BI-S)^{\frac{1}{2}}f, \quad \forall f \in \mathcal{H} \qquad (1)$$

Let $\{e_j\}_{j\in J}$ be an ONB for $\mathcal{H}$.

Then, by (1), and frame decomposition we have

$$Bf = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k + (BI-S)^{\frac{1}{2}} \sum_{j\in J} \langle (BI-S)^{\frac{1}{2}} f, e_j\rangle e_j$$

$$= \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k + \sum_{j \in J} \langle f, (B-IS)^{\frac{1}{2}} e_j \rangle (BI-S)^{\frac{1}{2}} e_j$$

$$\Rightarrow B\|f\|^2 = \langle Bf, f \rangle$$

$$= \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle$$

$$+ \sum_{j \in J} \langle f, (B-IS)^{\frac{1}{2}} e_j \rangle \langle (BI-S)^{\frac{1}{2}} e_j, f \rangle$$

$$= \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 + \sum_{j \in J} |\langle f, (B-IS)^{\frac{1}{2}} e_j \rangle|^2$$

$$\therefore \{ \phi_j \}_{j \in J} = \{ (B-IS)^{\frac{1}{2}} e_j \}_{j \in J}$$

is a seq. of vectors in $\mathcal{H}$ s.t.

$$\{f_k\}_{k=1}^{\infty} \cup \{\phi_j\}_{j \in J}$$

is a tight frame for $\mathcal{H}$ with bound B.

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→ <u>Theorem</u>: Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be Bessel sequences in $\mathcal{H}$. Then, $\exists$ Bessel sequences $\{p_j\}_{j\in J}$ and $\{v_j\}_{j\in J}$ in $\mathcal{H}$ s.t.

$$\{f_k\}_{k=1}^{\infty} \cup \{p_j\}_{j\in J} \quad \text{and} \quad \{g_k\}_{k=1}^{\infty} \cup \{v_j\}_{j\in J}$$

form a pair of dual frames for $\mathcal{H}$.

Proof: Let T and U denote the synthesis operators for $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. Since, $\mathcal{H}$ always admits a pair of dual frames. Let us denote the pair by $\{a_j\}_{j\in J} \cup \{b_j\}_{j\in J}$. Then, for any $f\in\mathcal{H}$, we have

$$f = UT^{*}f + (I - UT^{*})f$$

$$= \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k + \sum_{j\in J} \langle (I - UT^{*})f, a_j\rangle b_j$$

$$= \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k + \sum_{j\in J} \langle f, (I - UT^{*})^{*} a_j\rangle b_j \qquad \text{--- (1)}$$

The sequences $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$, $\{a_j\}_{j\in J}$ and $\{b_j\}_{j\in J}$ are Bessel sequences, and $\{(I-TU^*)a_j\}_{j\in J}$ is Bessel ($\because I-TU^*$ is bdd)

Thus, by lemma (*) and (1), we conclude that

$I=TU^*$
$\Leftrightarrow I=UT^*$
$\Leftrightarrow \langle f,g\rangle=\sum\langle f,f_k\rangle\langle g_k,\cdot\rangle$
are equivalent.

$\{f_k\}_{k=1}^{\infty}\cup\underbrace{\{(I-TU^*)a_j\}_{j\in J}} \rightarrow \{\varphi_k\}_{k\in J}$

and $\{g_k\}_{k=1}^{\infty}\cup\underbrace{\{b_j\}_{j\in J}} \rightarrow \{\psi_k\}_{k\in J}$

form a dual pair of frames for $\mathcal{H}$, as desired.

Q (1) Show that a frame $\{f_k\}_{k=1}^{\infty}$ is tight if and only if $\{f_k\}_{k=1}^{\infty}$ has a dual of the form $\{g_k\}_{k=1}^{\infty}=\{cf_k\}_{k=1}^{\infty}$ for some $c>0$.

Q (2): Let $\{e_k\}_{k=1}^{\infty}$ be an ONB for $\mathcal{H}$. Show that $\{e_k+e_{k+1}\}_{k=1}^{\infty}$ is a Bessel set but NOT a frame for $\mathcal{H}$. How you extend $\{e_k+e_{k+1}\}_{k}^{\infty}$ to a Tight frame for $\mathcal{H}$.

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Continuous frames

Defn: Let $\mathcal{H}$ be a complex Hilbert space and M a measure space with positive measure μ. A continuous frame for $\mathcal{H}$ is a family of vectors $\{f_k\}_{k\in M}$ for which:

1. for all $f \in \mathcal{H}$, the map
$$k \longmapsto \langle f, f_k \rangle$$
is measurable

2. $\exists\ 0 < A, B < \infty$ s.t.
$$A\|f\|^2 \le \int_M |\langle f, f_k\rangle|^2 d\mu \le B\|f\|^2 \quad \forall f \in \mathcal{H}. \qquad (\alpha)$$

Remark: If only upper inequality in (α) holds, then we say that $\{f_k\}_{k\in M}$ is a continuous Bessel family with Bessel bound B.

→ Remark: Let $\{f_k\}_{k\in M}$ be a continuous frame for $\mathcal{H}$. Then, by the Cauchy-Schwarz's inequality, the integral $\int_M \langle f, f_k\rangle\langle f_k, g\rangle\, d\mu$ is well defined for all $f, g \in \mathcal{H}$.

For a fixed $f \in \mathcal{H}$, the mapping

$$g \longrightarrow \int_M \langle f, f_k\rangle \langle f_k, g\rangle\, d\mu$$

is conjugate linear and bdd. Indeed, for any $g \in \mathcal{H}$, we have

$$\left|\int_M \langle f, f_k\rangle\langle f_k, g\rangle\, d\mu\right|^2 \leq \int_M |\langle f, f_k\rangle|^2 d\mu \times \int_M |\langle f_k, g\rangle|^2 d\mu$$

$$\leq B^2 \|f\|^2 \|g\|^2$$

Hence, by the Riesz representation theorem, $\exists$ a unique element - we call it $\int_M \langle f, f_k\rangle f_k\, d\mu$ s.t.

$$\left\langle \int_M \langle f, f_k\rangle f_k\, d\mu, g_k\right\rangle = \int_M \langle f, f_k\rangle \langle f_k, g\rangle\, d\mu$$

$\forall g \in \mathcal{H}$.

→ Let $\{f_k\}_{k\in M}$ be a continuous frame for $\mathcal{H}$. The operator $T : L^2(M,\mu) \to \mathcal{H}$ (in the weak sense) defined by

$$T : \{c_k\}_{k\in M} \longrightarrow \int_M c_k f_k \, d\mu$$

is called the pre-frame operator of $\{f_k\}_{k\in M}$. The pre-frame operator T is a bounded, linear operator, with adjoint $T^* : \mathcal{H} \to L^2(M,\mu)$ given by

$$T^* f \longrightarrow \{\langle f, f_k\rangle\}_{k\in M}$$

The frame operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ is given by

$$Sf \longrightarrow \int_M \langle f, f_k\rangle f_k \, d\mu$$

The frame operator S is bdd, linear and positive operator.

Note that $AI \leq S \leq BI$

Examples of continuous frames

first we discuss Operators on $L^2(\mathbb{R})$:—

1.) Translation operator, T_a

For $a \in \mathbb{R}$ $\quad T_a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

$$(T_a f)(x) = f(x-a) \quad , x \in \mathbb{R}.$$

2.) Modulation operator; E_b

for $b \in \mathbb{R}$ $\quad E_b : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

$$(E_b f)(x) = e^{2\pi i b x} f(x) \quad ; x \in \mathbb{R}$$

both these operators are W.D., linear, bdd, unitary

Well-defined Let $f \in L^2(\mathbb{R})$ $\quad$ T.S. $T_a f \in L^2(\mathbb{R})$

Consider, $\int_{-\infty}^{\infty} |T_a f(x)|^2 dx = \int_{-\infty}^{\infty} |f(x-a)|^2 dx$

$$= \int_{-\infty}^{\infty} |f(y)|^2 dy \quad \left(\begin{array}{l}\text{Let } y = x-a \\ dy = dx\end{array}\right)$$

$$< \infty$$

$\therefore T_a f \in L^2(\mathbb{R})$

Linear $\quad T_a(\alpha f + \beta g)(x) = (\alpha f + \beta g)(x-a)$

$$= \alpha f(x-a) + \beta g(x-a)$$

$$= \alpha T_a f(x) + \beta T_b g(x)$$

Bounded

$$\|T_a f\|^2 = \int_{-\infty}^{\infty} |T_a f(x)|^2 dx = \int_{-\infty}^{\infty} |f(x-a)|^2 dx$$

$$= \int_{-\infty}^{\infty} |f(y)|^2 dy \quad (\text{Let } y = x-a)$$

$$= \|f\|^2$$

unitary $\langle T_a f, g\rangle = \int_{-\infty}^{\infty} T_a f(x)\overline{g(x)}\,dx$

$$= \int_{-\infty}^{\infty} f(x-a)\overline{g(x)}\,dx$$

$$= \int_{-\infty}^{\infty} f(y)\overline{g(y+a)}\,dy \qquad (y = x-a)$$

$$= \int_{-\infty}^{\infty} f(y)\overline{T_{-a}g(y)}\,dy$$

$$= \langle f, T_{-a} g\rangle$$

$$\therefore T_a^* = T_{-a}$$

$$(T_a T_a^* f)(x) = f(x)$$

$$\Rightarrow T_a^{-1} = T_a^* = T_{-a}$$

Similarly, for modulation operator

→ Fourier Transform

For $f \in L^1(\mathbb{R})$, the Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\gamma}\,dx \quad , \gamma \in \mathbb{R}.$$

The ~~fourie~~ Fourier transform has an extension to a unitary operator on $L^2(\mathbb{R})$.

Definition 11.1.1 Fix a function $g \in L^2(\mathbb{R})\setminus\{0\}$. Furthermore, let $f \in L^2(\mathbb{R})$. The short-time Fourier transform of f w.r.t. g is defined as the function $\Psi_g(f)$ of two variables, given by

$$\Psi_g(f)(y,\gamma) = \int_{-\infty}^{\infty} f(x)\overline{g(x-y)}\, e^{-2\pi i x\gamma}\,dx \;;\; y, \gamma \in \mathbb{R}$$

Note Plancherel's equation :-

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle \quad \forall f, g \in L^2(\mathbb{R})$$
$$\& \quad \|\hat{f}\| = \|f\|$$

Proposition Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{g_1}(f_1)(a,b) \overline{\Psi_{g_2}(f_2)(a,b)} \, db \, da = \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle$$

i.e., $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f_1, E_b T_a g_1 \rangle \langle E_b T_a g_2, f_2 \rangle \, db \, da = \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle.$$

Proof Assume that $g_1, g_2 \in C_c(\mathbb{R})$ (space of all continuous f^n on $\mathbb{R}$ with compact support)

Consider for a fixed value of 'a'.

Define $F_1(x) = f_1(x) g_1(x-a)$.

Now, $$\Psi_{g_1}(f_1)(a,b) = \langle f_1, E_b T_a g_1 \rangle$$
$$= \int_{-\infty}^{\infty} f_1(x) e^{-2\pi i b x} \overline{g_1(x-a)} \, dx$$
$$= \hat{F}_1(b)$$

Similarly, define $F_2(x) = f_2(x) g_2(x-a)$

then $\Psi_{g_2}(f_2)(a,b) = \hat{F}_2(b)$

Consider,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{g_1}(f_1)(a,b) \overline{\Psi_{g_2}(f_2)(a,b)} \, db \, da = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}_1(b) \overline{\hat{F}_2(b)} \, db \, da$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(b) \overline{F_2(b)} \, db \, da \quad \text{(by Plancherel's equation)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(b)\,\overline{g_1(b-a) f_2(b)}\, g_2(b-a)\, db\, da$$ 14

$$= \int_{-\infty}^{\infty} f_1(b)\overline{f_2(b)} \left(\int_{-\infty}^{\infty} \overline{g_1(b-a)}\, g_2(b-a)\, da \right) db \quad \text{(by Fubini's Theorem)}$$

$$= \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle$$

~~Corollary~~:-

Corollary :- Let $g \in L^2(\mathbb{R}) \setminus \{0\}$. Then, $\{E_b T_a g\}_{a,b \in \mathbb{R}}$ is a continuous tight frame for $L^2(\mathbb{R})$ w.r.t. $M = \mathbb{R}^2$ equipped with the Lebesgue measure; the frame bound is $A = \|g\|^2$.

Proof Let $f_1 = f_2$ & $g_1 = g_2$ in above thm.

we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle f, E_b T_a g \rangle \langle E_b T_a g, f \rangle\, db\, da = \langle f, f \rangle \langle g, g \rangle$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\langle f, E_b T_a g \rangle|^2\, db\, da = \|f\|^2 \|g\|^2$$

$$\Rightarrow \int_M |\langle f, E_b T_a g \rangle|^2\, d\mu(a,b) = \|f\|^2 \|g\|^2$$

where $M = \mathbb{R}^2$.

$\therefore$ $\{E_b T_a g\}_{a,b \in \mathbb{R}}$ is a continuous tight frame for $L^2(\mathbb{R})$ w.r.t measure space $(\mathbb{R}^2, \mu)$ where μ = Lebesgue measure & frame bound is $A = \|g\|^2 \neq 0$ $(\because g \neq 0)$ $(A > 0)$

Gabor Frames in $L^2(\mathbb{R})$

Notations- for $a, b \in \mathbb{R}$.

$$T_a, E_b : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

$$T_a f(x) = f(x-a)$$

$$E_b f(x) = e^{2\pi i b x} f(x) \quad ; \quad x \in \mathbb{R}.$$

T_a is known as translation operator

E_b " " " modulation operator

Defn A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, where $a, b > 0$ & $g \in L^2(\mathbb{R})$ is a fixed fn.

i.e, For $g \in L^2(\mathbb{R})$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is s.t.b. a Gabor frame if $\exists$ $A, B > 0$ s.t.

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}).$$

Frames of this type are also called Weyl-Heisenberg frames. The function g is called the window function.

The Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ only involves translates with parameters na, $n \in \mathbb{Z}$ and modulations with parameters mb, $m \in \mathbb{Z}$. The points $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ form a so-called lattice in $\mathbb{R}^2$, and for this reason one frequently calls $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ a regular Gabor frame.

Let $\{(\mu_n, \lambda_n)\}_{n\in I}$ be an arbitrary countable subset of $\mathbb{R}^2$ & then we call $\{E_{\lambda_n}T_{\mu_n}g\}_{n\in I} = \{e^{2\pi i\lambda_n x} g(x-\mu_n)\}_{n\in\mathbb{Z}}$ an irregular Gabor frame.

Lemma- Let $f, g \in L^2(\mathbb{R})$ & $a, b > 0$ be given. Then, for any $n \in \mathbb{N}$, the following hold:

i) The series
$$\sum_{k \in \mathbb{Z}} f(x - k/b)\, \overline{g(x - na - k/b)}, \quad x \in \mathbb{R}.$$
converges absolutely for a.e. $x \in \mathbb{R}$.

ii) The mapping $x \longmapsto \sum_{k \in \mathbb{Z}} |f(x - k/b)\, \overline{g(x - na - k/b)}|$ belongs to $L^1(0, 1/b)$.

iii) The $1/b$ - periodic function $F_n \in L^1(0, 1/b)$ defined by
$$F_n(x) = \sum_{k \in \mathbb{Z}} f(x - k/b)\, \overline{g(x - na - k/b)}$$
has the fourier coefficients
$$c_m = b\, \langle f, E_{mb} T_{na} g \rangle, \quad m \in \mathbb{Z}.$$

Proof ii) Since $f, T_{na} g \in L^2(\mathbb{R})$.
$$\Rightarrow f\, \overline{T_{na} g} \in L^1(\mathbb{R}). \quad \text{(by Holder's inequality)}$$
$$\int_0^{1/b} \sum_{k \in \mathbb{Z}} |f(x - k/b)\, \overline{g(x - na - k/b)}|\, dx = \int_{-\infty}^{\infty} |f(x)\, \overline{g(x - na)}|\, dx \quad -(\#)$$
$$< \infty \quad \because f\, \overline{T_{na} g} \in L^1(\mathbb{R})$$
$$\Rightarrow x \longmapsto \sum_{k \in \mathbb{Z}} |f(x - k/b)\, g(x - na - k/b)| \text{ belongs to } L^1(0, 1/b).$$

$$\Big(\because \int_0^{1/b} \sum_{k \in \mathbb{Z}} |f(x - k/b)\, \overline{g(x - na - k/b)}|\, dx$$
$$= \dots + \int_0^{1/b} |f(x + 1/b)\, \overline{g(x - na + 1/b)}|\, dx + \int_0^{1/b} |f(x)\, \overline{g(x - na)}|\, dx +$$
$$\int_0^{1/b} |f(x - 1/b)\, \overline{g(x - na - 1/b)}|\, dx + \int_0^{1/b} |f(x - 2/b)\, \overline{g(x - na - 2/b)}|\, dx + \dots$$

Put $x - k/b = y \quad \forall k \qquad x = 0 \Rightarrow y = -k/b$
$$\Rightarrow dx = dy \qquad x = 1/b \Rightarrow y = (1-k)/b$$

$$= \cdots + \int_{1/b}^{2/b} |f(y)\overline{g(y-na)}|\,dy + \int_{0}^{1/b} |f(y)\overline{g(y-na)}|\,dy +$$

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$$\int_{-1/b}^{0} |f(y)\overline{g(y-na)}|\,dy + \int_{-2/b}^{-1/b} |f(y)\overline{g(y-na)}|\,dy + \cdots$$

$$= \int_{-\infty}^{\infty} |f(y)\overline{g(y-na)}|\,dy$$

$$= \int_{-\infty}^{\infty} |f(x)\overline{g(x-na)}|\,dx \quad \text{Replace } y \longrightarrow x\ )$$

i) $\therefore \int_{0}^{1/b} \sum_{k\in\mathbb{Z}} |f(x-k/b)\overline{g(x-na-k/b)}|\,dx < \infty$

$\Rightarrow \sum_{k\in\mathbb{Z}} |f(x-k/b)\overline{g(x-na-k/b)}|$ converges for a.e. $x \in [0, 1/b]$.

($\because$ If f is non-negative m'ble f^n
Then, $f = 0$ a.e. iff $\int f\,dx = 0$.
In general, $f = a$ a.e iff $\int f\,dx = a$ f.s. 'a')

And since $\sum_{k\in\mathbb{Z}} |f(x-k/b)\overline{g(x-na-k/b)}|$ is ~~peri~~ $1/b$- periodic

$\therefore$ it converges for a.e. $x \in \mathbb{R}$.

iii) Consider, $\langle f, E_{mb}T_{na}g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x-na)}\,e^{-2\pi i mbx}\,dx$

$$= \int_{0}^{1/b} \sum_{k\in\mathbb{Z}} f(x-k/b)\overline{g(x-na-k/b)}\,e^{-2\pi i mbx}\,dx$$

$$= \int_{0}^{1/b} F_n(x)\,e^{-2\pi i mbx}\,dx.$$

$\Rightarrow$ by the def^n of Fourier coefficients, F_n has the Fourier coefficients

$$c_m = b\,\langle f, E_{mb}T_{na}g\rangle, \quad m \in \mathbb{Z}$$

($\because$ $e_k(x) = b^{1/2} e^{2\pi i k b x}$, $k \in \mathbb{Z}$ forms an orthonormal basis for $L^2(0, 1/b)$ every $f \in L^2(0, 1/b)$ has an expansion

$$f = \sum_{k \in \mathbb{Z}} \langle f, e_k \rangle e_k$$

we usually expand 'f' in terms of $\{e^{2\pi i k b x}\}_{k \in \mathbb{Z}}$

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k b x}$$

where $c_k = b^{1/2} \langle f, e_k \rangle = b \int_0^{1/b} f(x) e^{-2\pi i k b x} dx$

so, for F_n $c_m = b \int_0^{1/b} F_n(x) e^{-2\pi i b m x} dx$

$$= b \langle f, E_{mb} T_{na} g \rangle$$

)

→ Necessary Conditions -

Thm-II Let $g \in L^2(\mathbb{R})$ & $a, b > 0$ be given. Then, the following hold -

i) If $ab > 1$, then $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$.

ii) If $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then

$$ab = 1 \iff \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} \text{ is a Riesz basis.}$$

Note: It is only possible for $\{E_{mb} T_{na} g\}_{m \in \mathbb{Z}}$ to be a frame if $ab \leq 1$. (by above result)

but the assumption $ab \leq 1$ is not enough for $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ to be a frame, even if $g \neq 0$.

For example - If $a \in]1/2, 1[$, $b = 1$, the f^{ns} $\{E_{mb} T_{na} \chi_{[0, 1/2]}\}_{m,n \in \mathbb{Z}}$ cannot form a frame since it is not complete in $L^2(\mathbb{R})$.

$\left(\because ab < 1 \qquad f_{m,n} = \left\{ e^{2\pi i m x} \chi_{[0,\frac{1}{2}]}(x - na) \right\}_{m,n \in \mathbb{Z}} \right.$ (3)

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Let $a = 0.6,\ n = 1,\ m = 0,\ f = \chi_{[0,\frac{1}{2}]}$.

then
$$\langle f_{m,n}, f \rangle = \int_{\mathbb{R}} f_{m,n}(x)\, \overline{f(x)}\, dx$$
$$= \int_{\mathbb{R}} e^{2\pi i m x} \chi_{[0,\frac{1}{2}]}(x - na)\, \chi_{[0,\frac{1}{2}]}(x)\, dx$$
$$= \int_{\mathbb{R}} \chi_{[0,\frac{1}{2}]}(x - 0.6)\, \chi_{[0,\frac{1}{2}]}(x)\, dx$$
$$= \int_0^{1/2} \chi_{[0,\frac{1}{2}]}(x - 0.6)\, dx$$
$$= 0 \qquad \left(\because x \in [0, \tfrac{1}{2}] \right)$$

$\therefore \exists\, 0 \neq f \in L^2(\mathbb{R})$ s.t. $\langle f_{m,n}, f \rangle = 0$ $\Big)$.